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Ostrowski type inequalities for functions whose derivatives are h-convex in absolute value

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Abstract

Some new inequalities of Ostrowski type for functions whose derivatives are h-convex in modulus are given. Applications for midpoint inequalities are provided as well.

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1 Introduction

1.1 Ostrowski Type Inequalities

Comparison between functions and integral means are incorporated in Ostrowski type inequalities as follows.

The first result in this direction is due to Ostrowski [38].

Theorem 1.1. Let $f:[a,b]\to\mathbb{R}$ be a differentiable function on (a,b) with the property that $|f'(t)|\le M$ for all $t\in(a,b)$. Then

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\le\left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right](b-a)M \tag{1.1}$$

for all $x\in[a,b]$.

The constant $\frac{1}{4}$ is the best possible in the sense that it cannot be replaced by a smaller quantity.

The following results for absolutely continuous functions hold (see [29] – [31]).

Theorem 1.2. Let $f:[a,b]\to\mathbb{R}$ be absolutely continuous on $[a,b]$. Then, for all $x\in[a,b]$, we have:

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$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \tag{1.2}$$
$$\leq \begin{cases} \left[\frac{1}{4} + \left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right](b-a)\|f'\|_\infty & \text{if } f' \in L_\infty[a,b]; \\ \frac{1}{(\alpha+1)^{\frac{1}{\alpha}}}\left[\left(\frac{x-a}{b-a}\right)^{\alpha+1} + \left(\frac{b-x}{b-a}\right)^{\alpha+1}\right]^{\frac{1}{\alpha}} \times (b-a)^{\frac{1}{\alpha}}\|f'\|_\beta & \text{if } f' \in L_\beta[a,b], \\ & \frac{1}{\alpha}+\frac{1}{\beta}=1, \\ & \alpha>1; \\ \left[\frac{1}{2} + \left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right]\|f'\|_1; \end{cases}$$

where $\|\cdot\|_{[a,b],r}$ $(r \in [1,\infty])$ are the usual Lebesgue norms on $L_r[a,b]$, i.e., we recall that

$$\|g\|_{[a,b],\infty} := ess \sup_{t\in[a,b]} |g(t)|$$

and

$$\|g\|_{[a,b],r} := \left(\int_a^b |g(t)|^r\,dt\right)^{\frac{1}{r}}, \ r \in [1,\infty).$$

The constants $\frac{1}{4}$, $\frac{1}{(p+1)^{\frac{1}{p}}}$ and $\frac{1}{2}$ respectively are sharp in the sense presented in Theorem 1.1.

The above inequalities can also be obtained from the Fink result in [33] on choosing $n = 1$ and performing some appropriate computations.

If one drops the condition of absolute continuity and assumes that f is Hölder continuous, then one may state the result (see for instance [21] and the references therein for earlier contributions):

Theorem 1.3. Let $f : [a,b] \to \mathbb{R}$ be of $r - H-$Hölder type, i.e.,

$$|f(x) - f(y)| \leq H|x-y|^r, \text{ for all } x, y \in [a,b], \tag{1.3}$$

where $r \in (0,1]$ and $H > 0$ are fixed. Then, for all $x \in [a,b]$, we have the inequality:

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \leq \frac{H}{r+1}\left[\left(\frac{b-x}{b-a}\right)^{r+1} + \left(\frac{x-a}{b-a}\right)^{r+1}\right](b-a)^r. \tag{1.4}$$

The constant $\frac{1}{r+1}$ is also sharp in the above sense.

Note that if $r = 1$, i.e., f is Lipschitz continuous, then we get the following version of Ostrowski's inequality for Lipschitzian functions (with L instead of H) (see for instance [13])

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \leq \left[\frac{1}{4} + \left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right](b-a)L, \tag{1.5}$$

where $x \in [a,b]$. Here the constant $\frac{1}{4}$ is also best.

Moreover, if one drops the condition of the continuity of the function, and assumes that it is of bounded variation, then the following result may be stated (see [15]).

Theorem 1.4. Assume that $f : [a,b] \to \mathbb{R}$ is of bounded variation and denote by $\bigvee_a^b (f)$ its total variation. Then

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)\, dt \right| \le \left[\frac{1}{2} + \left| \frac{x - \frac{a+b}{2}}{b-a} \right| \right] \bigvee_a^b (f) \tag{1.6}$$

for all $x \in [a,b]$. The constant $\frac{1}{2}$ is the best possible.

If we assume more about f, i.e., f is monotonically increasing, then the inequality (1.6) may be improved in the following manner [12] (see also the monograph [28]).

Theorem 1.5. Let $f : [a,b] \to \mathbb{R}$ be monotonic nondecreasing. Then for all $x \in [a,b]$, we have the inequality:

$$\begin{aligned} &\left| f(x) - \frac{1}{b-a} \int_a^b f(t)\, dt \right| \\ &\le \frac{1}{b-a} \left\{ [2x - (a+b)] f(x) + \int_a^b sgn(t-x) f(t)\, dt \right\} \\ &\le \frac{1}{b-a} \{ (x-a)[f(x) - f(a)] + (b-x)[f(b) - f(x)] \} \\ &\le \left[\frac{1}{2} + \left| \frac{x - \frac{a+b}{2}}{b-a} \right| \right] [f(b) - f(a)]. \end{aligned} \tag{1.7}$$

All the inequalities in (1.7) are sharp and the constant $\frac{1}{2}$ is the best possible.

The case for the convex functions is as follows [18]:

Theorem 1.6. Let $f : [a,b] \subset \mathbb{R} \to \mathbb{R}$ be a convex function on $[a,b]$. Then for any $x \in (a,b)$ one has the inequality

$$\begin{aligned} &\frac{1}{2} \left[(b-x)^2 f'_+(x) - (x-a)^2 f'_-(x) \right] \\ &\le \int_a^b f(t)\, dt - (b-a) f(x) \\ &\le \frac{1}{2} \left[(b-x)^2 f'_-(b) - (x-a)^2 f'_+(a) \right]. \end{aligned} \tag{1.8}$$

The constant $\frac{1}{2}$ is sharp in both inequalities. The second inequality also holds for $x = a$ or $x = b$.

For other Ostrowski's type inequalities for the Lebesgue integral, see [3]-[13] and [19].

Inequalities for the Riemann-Stieltjes integral may be found in [14], [16] while the generalization for isotonic functionals was provided in [17].

For the case of functions of self-adjoint operators on complex Hilbert spaces, see the recent monograph [20]

1.2 The Case of Derivatives that are Convex in Modulus

In [17], the author pointed out the following identity in representing an absolutely continuous function. Due to the fact that we use it throughout the paper we give here a short proof.

Lemma 1.7. Let $f : [a,b] \to \mathbb{R}$ be an absolutely continuous function on $[a,b]$. Then for any $x \in [a,b]$, one has the equality:

$$f(x) = \frac{1}{b-a}\int_a^b f(t)\,dt + \frac{1}{b-a}\int_a^b (x-t)\left(\int_0^1 f'[(1-\lambda)x+\lambda t]\,d\lambda\right)dt. \tag{1.9}$$

Proof. For any $t, x \in [a,b]$, $x \neq t$, one has

$$\frac{f(x)-f(t)}{x-t} = \frac{1}{x-t}\int_t^x f'(u)\,du = \int_0^1 f'[(1-\lambda)x+\lambda t]\,d\lambda,$$

showing that

$$f(x) = f(t) + (x-t)\int_0^1 f'[(1-\lambda)x+\lambda t]\,d\lambda \tag{1.10}$$

for any $t, x \in [a,b]$.

If we integrate (1.10) over t on $[a,b]$ and divide by $(b-a)$, we deduce the desired identity (1.9). Q.E.D.

Using the above lemma the following result can be pointed out improving Ostrowski's inequality [4].

Theorem 1.8. Let $f : [a,b] \to \mathbb{C}$ be an absolutely continuous function on $[a,b]$ so that $|f'|$ is convex on (a,b).

(i) If $f' \in L_\infty[a,b]$, then for any $x \in [a,b]$,

$$\begin{aligned}\left|f(x) - \frac{1}{b-a}\int_a^b f(t)dt\right| \\ \leq \frac{1}{2}\left[\frac{1}{4} + \left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right](b-a)\left[|f'(x)| + \|f'\|_\infty\right].\end{aligned} \tag{1.11}$$

The constant $\frac{1}{2}$ is sharp in the sense that it cannot be replaced by a smaller quantity.

(ii) If $f' \in L_p[a,b]$, $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, then for any $x \in [a,b]$,

$$\begin{aligned}\left|f(x) - \frac{1}{b-a}\int_a^b f(t)dt\right| \\ \leq \frac{1}{2(q+1)^{\frac{1}{q}}}\left[\left(\frac{b-x}{b-a}\right)^{q+1} + \left(\frac{x-a}{b-a}\right)^{q+1}\right]^{\frac{1}{q}}(b-a)^{\frac{1}{q}}\left\||f'(x)| + |f'|\right\|_p.\end{aligned} \tag{1.12}$$

The constant $\frac{1}{2}$ is sharp in the sense that it cannot be replaced by a smaller quantity.

(iii) If $f' \in L_1[a,b]$, then for any $x \in [a,b]$,

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)dt \right| \leq \frac{1}{2} \left[\frac{1}{2} + \left| \frac{x - \frac{a+b}{2}}{b-a} \right| \right] \left[(b-a) \left| f'(x) \right| + \left\| f' \right\|_1 \right]. \tag{1.13}$$

In order to extend this result for other classes of functions, we need the following preparatory section.

2 h-Convex Functions

2.1 Some Definitions

We recall here some concepts of convexities that are well known in the literature. Let I be an interval in $\mathbb{R}$.

Definition 2.1 ([32])**.** We say that $f : I \to \mathbb{R}$ is a Godunova-Levin function or that f belongs to the class $Q(I)$ if f is non-negative and for all $x, y \in I$ and $t \in (0,1)$ we have

$$f(tx + (1-t)y) \leq \frac{1}{t} f(x) + \frac{1}{1-t} f(y).$$

Some further properties of this class of functions can be found in [24], [25], [27], [37], [40] and [41]. Among others, its has been noted that non-negative monotone and non-negative convex functions belong to this class of functions.

Definition 2.2 ([27])**.** We say that a function $f : I \to \mathbb{R}$ belongs to the class $P(I)$ if it is nonnegative and for all $x, y \in I$ and $t \in [0,1]$ we have

$$f(tx + (1-t)y) \leq f(x) + f(y).$$

Obviously $Q(I)$ contains $P(I)$ and for applications it is important to note that also $P(I)$ contain all nonnegative monotone, convex and *quasi convex functions,* i. e. nonnegative functions satisfying

$$f(tx + (1-t)y) \leq \max\{f(x), f(y)\}$$

for all $x, y \in I$ and $t \in [0,1]$.

For some results on P-functions see [27] and [39] while for quasi convex functions, the reader can consult [26].

Definition 2.3 ([6])**.** Let s be a real number, $s \in (0,1]$. A function $f : [0,\infty) \to [0,\infty)$ is said to be s-convex (in the second sense) or Breckner s-convex if

$$f(tx + (1-t)y) \leq t^s f(x) + (1-t)^s f(y)$$

for all $x, y \in [0,\infty)$ and $t \in [0,1]$.

For some properties of this class of functions see [1], [2], [6], [7], [22], [23], [34], [35] and [43].

In order to unify the above concepts, S. Varošanec introduced the concept of h-convex functions as follows.

Assume that I and J are intervals in $\mathbb{R}, (0,1) \subseteq J$ and functions h and f are real non-negative functions defined in J and I, respectively.

Definition 2.4 ([46])**.** Let $h : J \to [0,\infty)$ with h not identical to 0. We say that $f : I \to [0,\infty)$ is an h-convex function if for all $x, y \in I$ we have

$$f(tx + (1-t)y) \leq h(t) f(x) + h(1-t) f(y)$$

for all $t \in (0,1)$.

For some results concerning this class of functions see [46], [5], [36], [44], [42] and [45].

2.2 Inequalities of Hermite-Hadamard Type

In [42] the authors proved the following Hermite-Hadamard type inequality for integrable h-convex functions.

Theorem 2.5. Assume that $f : I \to [0,\infty)$ is an h-convex function, $h \in L[0,1]$ and $f \in L[a,b]$ where $a, b \in I$ with $a < b$. Then

$$\frac{1}{2h\left(\frac{1}{2}\right)} f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(t)\,dt \leq [f(a) + f(b)] \int_0^1 h(t)\,dt. \tag{HH}$$

If we write (HH) for $h(t) = t$, then we get the classical Hermite-Hadamard inequality for convex functions.

If we write it for the case of P-type functions, i.e., $h(t) = 1$, then we get the inequality

$$\frac{1}{2} f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(t)\,dt \leq f(a) + f(b), \tag{2.1}$$

provided $f \in L[a,b]$, that has been obtained in [27].

If f is integrable on $[a,b]$ and Breckner s-convex on $[a,b]$, for $s \in (0,1)$, then by taking $h(t) = t^s$ in (HH) we get

$$2^{s-1} f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(t)\,dt \leq \frac{f(a) + f(b)}{s+1} \tag{2.2}$$

that was obtained in [22].

Since for the case of Godunova-Levin class of function we have $h(t) = \frac{1}{t}$, which is not Lebesgue integrable on $(0,1)$, we cannot apply the left inequality in (HH).

We can introduce now another class of functions.

Definition 2.6. We say that the function $f : I \to [0,\infty)$ is of s-Godunova-Levin type, with $s \in [0,1]$, if

$$f(tx + (1-t)y) \leq \frac{1}{t^s} f(x) + \frac{1}{(1-t)^s} f(y), \tag{2.3}$$

for all $t \in (0,1)$ and $x, y \in I$.

We observe that for $s=0$ we obtain the class of P-functions while for $s=1$ we obtain the class of Godunova-Levin. If we denote by $Q_s(I)$ the class of s-Godunova-Levin functions defined on I, then we obviously have

$$P(I)=Q_0(I)\subseteq Q_{s_1}(I)\subseteq Q_{s_2}(I)\subseteq Q_1(I)=Q(I)$$

for $0\le s_1\le s_2\le 1$.

We have the following Hermite-Hadamard type inequality.

Theorem 2.7. Assume that the function $f:I\to[0,\infty)$ is of s-Godunova-Levin type, with $s\in[0,1)$. If $f\in L[a,b]$ where $a,b\in I$ and $a<b$, then

$$\frac{1}{2^{s+1}}f\left(\frac{a+b}{2}\right)\le\frac{1}{b-a}\int_a^b f(t)\,dt\le\frac{f(a)+f(b)}{1-s}. \tag{2.4}$$

We notice that for $s=1$ the first inequality in (2.4) still holds and was obtained for the first time in [27].

3 Inequalities for Functions Whose Derivatives are h-Convex in Modulus

3.1 The Case of $|f'|$ is h-Convex

The following result holds:

Theorem 3.1. Let $f:[a,b]\to\mathbb{C}$ be an absolutely continuous function on $[a,b]$ so that $|f'|$ is h-convex on (a,b) with $h\in L[0,1]$.

(i) If $f'\in L_\infty[a,b]$, then for any $x\in[a,b]$,

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right| \tag{3.1}$$
$$\le\left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right](b-a)\left[|f'(x)|+\|f'\|_\infty\right]\int_0^1 h(t)\,dt.$$

(ii) If $f'\in L_p[a,b]$, $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, then for any $x\in[a,b]$,

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right| \tag{3.2}$$
$$\le\frac{1}{(q+1)^{\frac{1}{q}}}\left[\left(\frac{b-x}{b-a}\right)^{q+1}+\left(\frac{x-a}{b-a}\right)^{q+1}\right]^{\frac{1}{q}}\times(b-a)^{\frac{1}{q}}\left\||f'(x)|+|f'|\right\|_p\int_0^1 h(t)\,dt.$$

(iii) If $f'\in L_1[a,b]$, then for any $x\in[a,b]$,

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right|\le\left[\frac{1}{2}+\left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right]\times\left[(b-a)|f'(x)|+\|f'\|_1\right]\int_0^1 h(t)\,dt. \tag{3.3}$$

Proof. (i). Using (1.9) and taking the modulus, we have

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right| = \frac{1}{b-a}\left|\int_a^b\int_0^1 (x-t)\, f'\left[(1-\lambda)\, x+\lambda t\right] d\lambda dt\right|$$
$$\leq \frac{1}{b-a}\int_a^b\int_0^1 |x-t|\, |f'\left[(1-\lambda)\, x+\lambda t\right]|\, d\lambda dt := K$$

Utilizing the h-convexity of $|f'|$ we have

$$\begin{aligned}
K &\leq \frac{1}{b-a}\int_a^b\int_0^1 |x-t|\left[h(1-\lambda)\,|f'(x)|+h(\lambda)\,|f'(t)|\right] d\lambda dt\\
&= \frac{1}{b-a}\int_a^b |x-t|\left[|f'(x)|\int_0^1 h(1-\lambda)\, d\lambda+|f'(t)|\int_0^1 h(\lambda)\, d\lambda\right] dt\\
&= \frac{1}{b-a}\int_0^1 h(\lambda)\, d\lambda\int_a^b |x-t|\left[|f'(x)|+|f'(t)|\right] dt := M(x)\int_0^1 h(\lambda)\, d\lambda\\
&\leq \frac{1}{b-a}\int_0^1 h(\lambda)\, d\lambda \; ess\sup_{t\in[a,b]}\left[|f'(x)|+|f'(t)|\right]\int_a^b |x-t|\, dt\\
&= \left[\frac{(x-a)^2+(b-x)^2}{2(b-a)}\right]\left[|f'(x)|+\|f'\|_\infty\right]\int_0^1 h(\lambda)\, d\lambda\\
&= \left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right](b-a)\left[|f'(x)|+\|f'\|_\infty\right]\int_0^1 h(\lambda)\, d\lambda,
\end{aligned}$$

for any $x\in[a,b]$, and the inequality (3.1) is proved.

(ii). As above, we have

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right| \leq \frac{1}{b-a}\int_a^b |x-t|\left[|f'(x)|+|f'(t)|\right] dt := M(x)\int_0^1 h(\lambda)\, d\lambda.$$

Using Hölder's integral inequality for $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, we get that

$$\begin{aligned}
M(x) &\leq \frac{1}{b-a}\left(\int_a^b |x-t|^q\, dt\right)^{\frac{1}{q}}\left(\int_a^b \left(|f'(x)|+|f'(t)|\right)^p dt\right)^{\frac{1}{p}}\\
&= \frac{1}{b-a}\left[\frac{(b-x)^{q+1}+(x-a)^{q+1}}{q+1}\right]^{\frac{1}{q}}\left\||f'(x)|+|f'|\right\|_p
\end{aligned}$$

and the inequality (3.2) is proved.

(iii). We also have that

$$M(x) \leq \sup_{t\in[a,b]} |x-t| \frac{1}{b-a} \int_a^b [|f'(x)| + |f'(t)|] \, dt$$
$$= \frac{1}{b-a)} \max(x-a, b-x) \left[(b-a) |f'(x)| + \int_a^b |f'(t)| \, dt \right]$$
$$= \left[\frac{1}{2} + \left| \frac{x - \frac{a+b}{2}}{b-a} \right| \right] [(b-a) |f'(x)| + \|f'\|_1]$$

and the inequality (3.3) is proved. Q.E.D.

The following particular case is interesting.

Corollary 3.2. With the assumptions of Theorem 3.1, we have the midpoint inequality

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)dt \right| \tag{3.4}$$
$$\leq \frac{1}{4}(b-a) \left[\left| f'\left(\frac{a+b}{2}\right) \right| + \|f'\|_\infty \right] \int_0^1 h(t) \, dt,$$

provided $f' \in L_\infty[a,b]$.

If $f' \in L_p[a,b]$, $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, then, we have,

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)dt \right| \tag{3.5}$$
$$\leq \frac{1}{2}(b-a)^{\frac{1}{q}} \left(\int_a^b \left[\left| f'\left(\frac{a+b}{2}\right) \right| + |f'(t)| \right]^p dt \right)^{\frac{1}{p}} \int_0^1 h(t) \, dt.$$

If $f' \in L_1[a,b]$, then

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)dt \right| \tag{3.6}$$
$$\leq \frac{1}{2} \left[(b-a) \left| f'\left(\frac{a+b}{2}\right) \right| + \int_a^b |f'(t)| \, dt \right] \int_0^1 h(t) \, dt.$$

Remark 3.3. We observe that if $|f'|$ is convex on (a,b), then Theorem 3.1 reduces to Theorem 1.8.

Assume that $|f'|$ is Breckner s-convex on $[a,b]$, for $s \in (0,1)$.

(a) If $f' \in L_\infty[a,b]$, then for any $x \in [a,b]$,

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt \right| \leq \frac{1}{s+1}\left[\frac{1}{4} + \left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right](b-a)\left[|f'(x)| + \|f'\|_\infty\right]. \tag{3.7}$$

(aa) If $f' \in L_p[a,b]$, $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, then for any $x \in [a,b]$,

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt \right| \leq \frac{1}{(s+1)(q+1)^{\frac{1}{q}}}\left[\left(\frac{b-x}{b-a}\right)^{q+1} + \left(\frac{x-a}{b-a}\right)^{q+1}\right]^{\frac{1}{q}} \times (b-a)^{\frac{1}{q}} \, \||f'(x)| + |f'|\|_p . \tag{3.8}$$

(aaa) If $f' \in L_1[a,b]$, then for any $x \in [a,b]$,

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt \right| \leq \frac{1}{s+1}\left[\frac{1}{2} + \left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right] \times \left[(b-a)|f'(x)| + \|f'\|_1\right]. \tag{3.9}$$

Assume that $|f'|$ is of s-Godunova-Levin type, with $s \in [0,1)$.

(b) If $f' \in L_\infty[a,b]$, then for any $x \in [a,b]$,

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt \right| \leq \frac{1}{1-s}\left[\frac{1}{4} + \left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right](b-a)\left[|f'(x)| + \|f'\|_\infty\right]. \tag{3.10}$$

(bb) If $f' \in L_p[a,b]$, $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, then for any $x \in [a,b]$,

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt \right| \leq \frac{1}{(1-s)(q+1)^{\frac{1}{q}}}\left[\left(\frac{b-x}{b-a}\right)^{q+1} + \left(\frac{x-a}{b-a}\right)^{q+1}\right]^{\frac{1}{q}} \times (b-a)^{\frac{1}{q}} \, \||f'(x)| + |f'|\|_p . \tag{3.11}$$

(bbb) If $f' \in L_1[a,b]$, then for any $x \in [a,b]$,

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt \right| \leq \frac{1}{1-s}\left[\frac{1}{2} + \left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right] \times \left[(b-a)|f'(x)| + \|f'\|_1\right]. \tag{3.12}$$

3.2 The Case of $|f'|^p$ is h-Convex

The following result also holds:

Theorem 3.4. Let $f:[a,b]\to\mathbb{C}$ be an absolutely continuous function on $[a,b]$ so that $|f'|^p$ with $p>1$ is h-convex on (a,b) and $h\in L[0,1]$.

(i) If $f'\in L_\infty[a,b]$, then for any $x\in[a,b]$,

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right| \le \left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right](b-a)\times\left[|f'(x)|^p+\|f'\|_\infty^p\right]^{1/p}\left(\int_0^1 h(t)\,dt\right)^{1/p}. \tag{3.13}$$

(ii) If $f'\in L_p[a,b]$, $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, then for any $x\in[a,b]$,

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right| \le \frac{(b-a)^{\frac{1}{q}}}{(q+1)^{1/q}}\left[\left(\frac{b-x}{b-a}\right)^{q+1}+\left(\frac{x-a}{b-a}\right)^{q+1}\right]^{1/q}\times\left[(b-a)\left|f'(x)\right|^p+\|f'\|_p^p\right]^{1/p}\left(\int_0^1 h(t)\,dt\right)^{1/p}. \tag{3.14}$$

(iii) If $f'\in L_p[a,b]$, then for any $x\in[a,b]$,

$$\begin{aligned}\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right| &\le \left[\frac{1}{2}+\left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right]\times\left\|\left|f'(x)\right|^p+|f'|^p\right\|^p\left(\int_0^1 h(t)\,dt\right)^{1/p} \\ &\le \left[\frac{1}{2}+\left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right]\times\left((b-a)\left|f'(x)\right|^p+\|f'\|_p^p\right)^{1/p}\left(\int_0^1 h(t)\,dt\right)^{1/p}.\end{aligned} \tag{3.15}$$

Proof. As in the proof of Theorem 3.1 we have

$$\begin{aligned}\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right| &= \frac{1}{b-a}\left|\int_a^b\int_0^1 (x-t)\,f'\left[(1-\lambda)x+\lambda t\right]d\lambda dt\right| \\ &\le \frac{1}{b-a}\int_a^b |x-t|\left(\int_0^1 \left|f'\left[(1-\lambda)x+\lambda t\right]\right|d\lambda\right)dt := K\end{aligned}$$

for any $x\in[a,b]$.

By Hölder's integral inequality we have

$$\begin{aligned}\int_0^1 \left|f'\left[(1-\lambda)x+\lambda t\right]\right|d\lambda &\le \left(\int_0^1 1^q d\lambda\right)^{1/q}\left(\int_0^1 \left|f'\left[(1-\lambda)x+\lambda t\right]\right|^p d\lambda\right)^{1/p} \\ &= \left(\int_0^1 \left|f'\left[(1-\lambda)x+\lambda t\right]\right|^p d\lambda\right)^{1/p}\end{aligned}$$

for any $x \in [a, b]$, where $\frac{1}{p} + \frac{1}{q} = 1$, $p > 1$.

Since $|f'|^p$ is h-convex on (a, b) with $h \in L[0, 1]$, then

$$\int_0^1 |f'[(1-\lambda)x + \lambda t]|^p d\lambda \leq \left[|f'(x)|^p + |f'(t)|^p\right] \int_0^1 h(\lambda) d\lambda,$$

for any $x \in [a, b]$.

Therefore

$$K \leq \frac{1}{b-a} \left(\int_0^1 h(\lambda) d\lambda\right)^{1/p} \int_a^b |x - t| \left[|f'(x)|^p + |f'(t)|^p\right]^{1/p} dt \tag{3.16}$$

for any $x \in [a, b]$.

(i). Now, if $f' \in L_\infty [a, b]$ then

$$\begin{aligned}
&\int_a^b |x - t| \left[|f'(x)|^p + |f'(t)|^p\right]^{1/p} dt \\
&\leq ess \sup_{t \in [a,b]} \left[|f'(x)|^p + |f'(t)|^p\right]^{1/p} \int_a^b |x - t| dt \\
&= \left[|f'(x)|^p + \|f'\|_\infty^p\right]^{1/p} \frac{1}{2} \left[(x-a)^2 + (b-x)^2\right]
\end{aligned}$$

for any $x \in [a, b]$, and utilizing (3.16), the inequality (3.13) is proved.

(ii). If $f' \in L_p[a, b]$, $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, then by Hölder's inequality we have

$$\begin{aligned}
&\int_a^b |x - t| \left[|f'(x)|^p + |f'(t)|^p\right]^{1/p} dt \\
&\leq \left(\int_a^b |x - t|^q dt\right)^{1/q} \left(\int_a^b \left(\left[|f'(x)|^p + |f'(t)|^p\right]^{1/p}\right)^p dt\right)^{1/p} \\
&= \left[\frac{(b-x)^{q+1} + (x-a)^{q+1}}{q+1}\right]^{1/q} \left[(b-a)|f'(x)|^p + \|f'\|_p^p\right]^{1/p} \\
&= \frac{(b-a)^{1+\frac{1}{q}}}{(q+1)^{1/q}} \left[\left(\frac{b-x}{b-a}\right)^{q+1} + \left(\frac{x-a}{b-a}\right)^{q+1}\right]^{1/q} \times \left[(b-a)|f'(x)|^p + \|f'\|_1^p\right]^{1/p}
\end{aligned}$$

for any $x \in [a, b]$, and by (3.16) we deduce the desired inequality (3.14).

(iii). If $f' \in L_p[a,b]$, then by Hölder's inequality we also have

$$\begin{aligned}
&\int_a^b |x-t| \left[|f'(x)|^p + |f'(t)|^p \right]^{1/p} dt \\
&\le \sup_{t\in[a,b]} |x-t| \int_a^b \left[|f'(x)|^p + |f'(t)|^p \right]^{1/p} dt \\
&= \max\{x-a, b-x\} \int_a^b \left[|f'(x)|^p + |f'(t)|^p \right]^{1/p} dt \\
&= (b-a) \left[\frac{1}{2} + \left| \frac{x - \frac{a+b}{2}}{b-a} \right| \right] \left\| |f'(x)|^p + |f'|^p \right\|^p \\
&\le (b-a) \left[\frac{1}{2} + \left| \frac{x - \frac{a+b}{2}}{b-a} \right| \right] \left(\int_a^b \left[|f'(x)|^p + |f'(t)|^p \right] dt \right)^{1/p} \\
&= (b-a) \left[\frac{1}{2} + \left| \frac{x - \frac{a+b}{2}}{b-a} \right| \right] \left((b-a) |f'(x)|^p + \|f'\|_p^p \right)^{1/p}
\end{aligned}$$

for any $x \in [a,b]$. Q.E.D.

The following midpoint type inequalities are of interest.

Corollary 3.5. With the assumptions of Theorem 3.4, we have the inequality

$$\begin{aligned}
&\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(t)dt \right| \qquad (3.17)\\
&\le \frac{1}{4}(b-a) \left[\left| f'\left(\frac{a+b}{2}\right) \right|^p + \|f'\|_\infty^p \right]^{1/p} \left(\int_0^1 h(t)\,dt \right)^{1/p},
\end{aligned}$$

provided $f' \in L_\infty[a,b]$.

If $f' \in L_p[a,b]$, $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, then we have

$$\begin{aligned}
&\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(t)dt \right| \qquad (3.18)\\
&\le \frac{1}{2(q+1)^{1/q}} (b-a)^{\frac{1}{q}} \times \left[(b-a) \left| f'\left(\frac{a+b}{2}\right) \right|^p + \|f'\|_p^p \right]^{1/p} \left(\int_0^1 h(t)\,dt \right)^{1/p}.
\end{aligned}$$

If $f' \in L_p[a,b]$, then

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)dt \right| \tag{3.19}$$
$$\leq \frac{1}{2}\left\| \left| f'\left(\frac{a+b}{2}\right)\right|^p + |f'|^p \right\|^p \left(\int_0^1 h(t)\,dt\right)^{1/p}$$
$$\leq \frac{1}{2}\left((b-a)\left| f'\left(\frac{a+b}{2}\right)\right|^p + \|f'\|_p^p \right)^{1/p} \left(\int_0^1 h(t)\,dt\right)^{1/p}.$$

Remark 3.6. The interested reader can state the corresponding particular inequalities for different h-convex functions. However the details are omitted.

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